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A Note on Fuzzy Observables

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Abstract

This paper is concerned with positive-operator-valued measures that are generated by modeling quantum measurements in which it is not possible to avoid residual experimental errors. A positive-operator-valued measure is constructed that has the same statistical properties as an observable defined on a Hilbert space that has a straightforward probabilistic interpretation.

1. *Introduction*

Positive-operator-valued measures, often also called generalized observables, have been introduced during the last years both in theoretical physics (see, for example, Jauch and Piron, 1967; Twareque Ali and Emch, 1974; Davies and Lewis, 1970; Benioff, 1972) and in some fields of applications, like the theory of quantum communications channels and in the theory of optimal receivers for optical signals (see Helstrom, 1970; Holevo, 1973; and others).

The main characteristic of generalized observables seems to be that, by the Naimark theorem (Akhiezer and Glazman, 1963), they are in some way equivalent to observables defined on an appropriate extended Hilbert space. To give a physical meaning to such an extended space is a critical, and yet unsolved, point for the applications and also represents a challenging theoretical problem.

It is also worth noticing that all authors on this subject seem to agree on the fact that it is not possible, in general, to provide a physical interpretation of the Naimark theorem. The present paper is concerned with positive-operator-valued measures that are generated by modeling "actual quantum measurements," i.e., quantum measurements in which it is not possible to avoid residual experimental errors.

A simple model for the experimental uncertainty is assumed which leads to a positive-operator-valued measure that has the same statistical properties on an observable defined on the tensor product between the Hilbert space associated with the system under consideration and a Hilbert space which can be interpreted as the representation of a wide class of experimental disturbances.

2. *Modeling Experimental Uncertainty*

Let us consider the observable X of some quantum physical system. By the theory of quantum mechanics X is represented by a projection-valued measure $P_X(\cdot)$ defined on some separable Hilbert space H which represents the system.

The physical meaning of that is the following: If an *ideal* measurement of X is performed, then the probability that the observed value of X belongs to some Borel set E of the real line is, given the pure state ψ,

$$p_X(E/\psi) = \langle\psi|P_X(E)\psi\rangle_H = \mathrm{Tr}_H P^{\psi} P_X(E) \tag{2.1}$$

where P^{ψ} is the orthogonal projection on the one-dimensional space spanned by ψ, or, given the general state ρ

$$p_X(E/\rho) = \mathrm{Tr}_H \rho P_X(E) \tag{2.2}$$

It is apparent that the probability measure $p_X(\cdot/\psi)$ has only a theoretical meaning in the sense that it does not correspond in general to the actual experimental situations.

The problem that is considered here is how to construct a new probability measure $\tilde{p}_X(\cdot/\psi)$ that takes into account the experimental errors.

In previous works (Janch and Piron, 1967; Twareque Ali and Emch, 1974) the probability $\tilde{p}(E_{x_0}/\psi)$, where E_{x_0} is an interval having center in x_0, is calculated by modeling such a fuzziness with an uncertainty in the localization of the midpoint x_0.

It is assumed that the observed midpoint x' is distributed on the real line R according to some probability density $f_{x_0}(\cdot)$ which is supposed symmetric around x_0.

The probability $\tilde{p}(E_{x_0}/\psi)$ is then calculated by averaging on all possible x' in the following way:

$$\tilde{p}(E_{x_0}/\psi) = \int_R \langle\psi|P_X(E_{x'})\psi\rangle f_{x_0}(x')dx' = \langle\psi|\mathscr{A}(E_{x_0})\psi\rangle \tag{2.3}$$

This construction leads to the positive-operator-valued measure defined for every interval E_{x_0} on the real line, by

$$\mathscr{A}(E_{x_0}) = \int_R P_X(E_{x'})f_{x_0}(x')dx' \tag{2.4}$$

Following a different line of thinking we simply model here the "observed value of a fuzzy measurement" as the sum of the "observed value of an ideal measurement" plus an error ϵ.

Calling y' the observed value of the imprecise measurement of X, we define

$$y' = y + \epsilon \tag{2.5}$$

where y represents the ideal observed value of X and ϵ is a real random variable distributed according to the probability density f_0. The function f_0 is supposed symmetric around zero.

If y and ϵ are regarded as two independent random variables having density probability, respectively, $g_\psi(\cdot)$ and $f_0(\cdot)$, the elementary probability rules give for the probability density of y'

$$h_\psi(y') = \int_R g_\psi(y) f_0(y' - y) dy \tag{2.6}$$

In our case we have, by definition,

$$g_\psi(y)dy \;\underline{\Delta}\; \langle \psi | P_x(dy) \psi \rangle \tag{2.7}$$

and then

$$h_\psi(y') = \int_R f_0(y' - y) \langle \psi | P_x(dy) \psi \rangle \tag{2.8}$$

and, for any Borel set A on the real line

$$\tilde{p}_x(A/\psi) = \int_R \int_R \chi_A(y') f_0(y' - y) dy' \langle \psi | P_x(dy) \psi \rangle = \langle \psi | Q(A) \psi \rangle \tag{2.9}$$

where $Q(\cdot)$ is the positive-operator-valued measure defined by

$$Q(A) = \int_R \int_R \chi_A(y') f_0(y' - y) dy' P_x(dy), \qquad \forall A \in \mathscr{B}(R) \tag{2.10}$$

It can be interesting to compare the generalized observable Q so constructed with that defined in (2.4).

For this purpose we define the conditional probability

$$p(A/y) = \int_R \chi_A(y') f_0(y' - y) dy', \qquad A \in \mathscr{B}(R) \tag{2.11}$$

We can write, for any interval E_{x_0}

$$Q(E_{x_0}) = \int_R p(E_{x_0}/y) P_x(dy) \tag{2.12}$$

Now, by observing that, by the symmetry of f_0

$$f_0(y' - y) = f_y(y') = f_{y'}(y) \tag{2.13}$$

where $f_y(\cdot)$ is the function f_0 shifted in order to have its center in y, we can write

$$p(E_{x_0}/y) = \int_{E_{x_0}} f_y(y') dy' = \int_{E_y} f_{x_0}(y') dy' = p(E_y/x_0) \tag{2.14}$$

Now, calling $l_{E_{x_0}} = l_{E_y}$ the length of the interval E_{x_0} and noticing that, from (2.11) and (2.13),

$$\int_R p(E_{x_0}/y)dy = l_{E_{x_0}} = l_{E_y} \tag{2.15}$$

We can define the density probability function

$$f'_{x_0}(y) \triangleq p(E_y/x_0)/l_{E_y} \tag{2.16}$$

and therefore $Q(E_{x_0})$ can be written as

$$Q(E_{x_0}) = \int_R f'_{x_0}(y) l_{E_y} P_x(dy) \tag{2.17}$$

One can see that $P_x(E_y)dy$ in (2.4) has been here substituted by $l_{E_y}P_x(dy)$, thus interchanging the role of the Lebesgue measure with that of the projection-valued measure P_x.

3. *Fuzzy Measurements as Observables on Extended Spaces*

From the Naimark theorem (Akhiezer and Glazman, 1963) we know that, given any positive-operator-valued measure $Q(\cdot)$ defined on the Hilbert space H it is always possible to find an extended space H and a projection-valued measure $\tilde{P}(\cdot)$ such that

$$\text{(i)}\ H = P_H \tilde{H} \tag{3.1}$$

$$\text{(ii)}\ Q(E) = P_H \tilde{P}(E) P_H \tag{3.2}$$

A straightforward consequence of the Naimark theorem is the following proposition (Holevo, 1973).

Proposition 1. Let $Q(\cdot)$ be a positive-operator-valued measure on H. Then there exists a Hilbert space H_0, a state ρ_0 on H_0 and a projection-valued measure $\tilde{P}(\cdot)$ on the tensor product $H_0 \otimes H$ such that

$$\mathrm{Tr}_H \rho Q(E) = \mathrm{Tr}_{H_0 \otimes H}(\rho_0 \otimes \rho)\tilde{P}(E), \qquad E \in \mathscr{B}(R) \tag{3.3}$$

for each state ρ on H.

The scope of this section is to identify in the most natural way a triple $\{H_0, \rho_0, \tilde{P}(\cdot)\}$ which verify Proposition 1 for the positive-operator-valued measure defined in (2.10). Let us consider again the conditional probability $p(\cdot/y)$ defined by (2.11). It is immediate to check that

$$\begin{aligned} p(A/y) &= \int_R \chi_A(y+\epsilon) f_0(\epsilon) d(y+\epsilon) \\ &= \int_R \chi_{A_y}(\epsilon) f_0(\epsilon) d\epsilon, \qquad A \in \mathscr{B}(R) \end{aligned} \tag{3.4}$$

for every fixed y and A_y being the Borel set obtained by shifting all the points of A by y.

Let now $\mathscr{L}^2$ be the set of all absolutely square integrable complex valued functions on R. As is well known, $\mathscr{L}^2$ is a separable Hilbert space with respect to the inner product

$$\langle f|g\rangle_{\mathscr{L}^2} = \int_R f(\xi)\overline{g(\xi)}d\xi, \qquad f \in \mathscr{L}^2, \qquad g \in \mathscr{L}^2 \tag{3.5}$$

Therefore, defined on $\mathscr{L}^2$ the projection operator

$$\Xi(A_y)f = \chi_{A_y}\cdot f, \qquad f \in \mathscr{L}^2 \tag{3.6}$$

we can write

$$p(A/y) = \langle f_0^{1/2}|\Xi(A_y)f_0^{1/2}\rangle_{\mathscr{L}^2} = \mathrm{Tr}_{\mathscr{L}^2} P^{f_0^{1/2}}\,\Xi(A_y) \tag{3.7}$$

and then $Q(\cdot)$ takes the form

$$Q(A) = \int_R [\mathrm{Tr}_{\mathscr{L}^2} P^{f_0^{1/2}}\,\Xi(A_y)]\, P_x(dy) \tag{3.8}$$

Now, given any state ρ on H, the properties of the trace and of the tensor product spaces immediatly give

$$\mathrm{Tr}_H \rho Q(A) = \int_R \mathrm{Tr}_{\mathscr{L}^2} P^{f_0^{1/2}}\,\Xi(A_y)\cdot \mathrm{Tr}_H \rho P_x(dy)$$

$$= \mathrm{Tr}_{\mathscr{L}^2\otimes H}(P^{f_0^{1/2}}\otimes\rho)\int_R \Xi(A_y)\otimes P_x(dy) \tag{3.9}$$

Defined

$$\tilde{P}(A) = \int_R \Xi(A_y)\otimes P_x(dy) \tag{3.10}$$

one can see that the triple $\{\mathscr{L}^2, P^{f_0^{1/2}}, \tilde{P}(A)\}$ satisfies Proposition 1 for the positive-operator-valued measure $Q(\cdot)$ provided $\tilde{P}(A)$ is a projection-valued measure defined on $\mathscr{L}^2 \otimes H$. It remains then only to check the three following properties.

(i) $\tilde{P}(A)$ is a projection defined on $\mathscr{L}^2 \otimes H$, $\forall A \in \mathscr{B}(R)$
(ii) If A and B are any two disjoint Borel sets then

$$\tilde{P}(A)\tilde{P}(B) = \mathbb{O}_{\mathscr{L}^2\otimes H} \tag{3.11}$$

(iii) If $\{A^i\}_{i=0}^{\infty}$ is any family of Borel sets such that $\cup_i A^i = R$ and $A^i \cap A^j = \phi$ if $i \neq j$ then

$$\sum_i \tilde{P}(A_i) = \mathbb{I}_{\mathscr{L}^2\otimes H} \tag{3.12}$$

As is well known, the right-hand side of (3.10) is by definition the limit of Rieman–Stieltjes sums of the type

$$\sum_i \Xi(A_{y_i}) \otimes P_x(\Delta_{y_i}) \tag{3.13}$$

where Δ_{y_i} are intervals such that $\cup_i \Delta_{y_i} = R$, $\Delta_{y_i} \cap \Delta_{y_j} = \phi$ if $i \neq j$ and the limit is done by taking the length of the intervals Δ_{y_i} smaller and smaller. The properties (i) and (ii) of $\tilde{P}(A)$ follow from the analogous properties of the operators defined by (3.13). In fact it is immediate to prove that $\Xi(A_{y_i}) \otimes P_x(\Delta_{y_i})$ is a projection $\forall i$.

In addition we have that, if $i \neq j$,

$$\begin{aligned}[\Xi(A_{y_i}) \otimes P_x(\Delta_{y_i})]\, [\Xi(A_{y_j}) \otimes P_x(\Delta_{y_j})] &= \Xi(A_{y_i})\Xi(A_{y_j}) \otimes \mathbb{O}_H \\ = \mathbb{O}_{\mathscr{L}^2 \otimes H} = [\Xi(A_{y_j}) \otimes P_x(\Delta_{y_j})]\, &[\Xi(A_{y_i}) \otimes P_x(\Delta_{y_i})]\end{aligned} \tag{3.14}$$

Therefore, since the sum of two projections is a projection if and only if they commute, (3.14) implies that any sum of the type (3.13) is a projection, and then property (i) is satisfied. [One could notice that $\bar{P}(A \times B) \triangleq \Xi(A) \otimes P_x(B)$ defines a projection-valued measure on $\mathscr{B}(R^2)$.]

To prove property (ii) let us consider, for two disjoint Borel sets A and B, the product

$$\begin{aligned}\{\sum_i \Xi(A_{y_i}) \otimes P_x(\Delta_{y_i})\}\{\sum_j \Xi(B_{y_j}) \otimes P_x(\Delta_{y_j})\} \\ = \sum_{ij} \Xi(A_{y_i})\Xi(B_{y_j}) \otimes P_x(\Delta_{y_i})P_x(\Delta_{y_j})\end{aligned} \tag{3.15}$$

We have

$$P_x(\Delta_{y_i})P_x(\Delta_{y_j}) = \mathbb{O}_H \qquad \text{if } i \neq j \tag{3.16}$$

and

$$\Xi(A_{y_i})\Xi(B_{y_j}) = \mathbb{O}_{\mathscr{L}^2} \qquad \text{if } i = j \tag{3.17}$$

since $A \cap B = \phi$ implies $A_{y_i} \cap B_{y_i} = \phi$, both A and B being shifted by y_i. The product in (3.15) is then always zero.

Finally, given any family of Borel sets $\{A^\kappa\}_{\kappa=0}^{\infty}$ such that $\cup_\kappa A^\kappa = R$ and $A^\kappa \cap A^J = \phi$ if $\kappa \neq J$ one can immediately check that

$$\sum_\kappa \tilde{P}(A_\kappa) = \sum_\kappa \int_R \Xi(A_{y^\kappa}) \otimes P_x(dy) = \int_R \mathbb{1}_{\mathscr{L}^2} \otimes P_x(dy) = \mathbb{1}_{\mathscr{L}^2} \otimes \mathbb{1}_H \tag{3.18}$$

and therefore property (iii) is also proved.

4. *Conclusions*

By modeling in a simple way the experimental uncertainty a positive-operator-valued measure has been constructed that has the same statistical properties as the projection-valued measure $\tilde{P}(\cdot)$, provided that for the disturbance ϵ (supposed independent from the "ideal observed value"!) the state $\rho_0 = P^{f_0^{1/2}}$ is given.

The probability density function f_0, which gives the statistical properties of the disturbance, is only required to be symmetric and Lebesgue integrable.

As a consequence its square root belongs to the $\mathscr{L}^2$ space, which plays here the role of "the separable Hilbert space representing the disturbance."

The unit norm vector $f_0^{1/2}$ represents "the (pure) state of the disturbance" in the usual quantum notation.

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